COMPETITION CORNER

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PROBLEMS

1. (Bulgaria 1994) A triangle ABC is given with $p = \frac{1}{2}(AB + BC + CA)$. A circle k_1 touches the side BC and the sides AB, AC extended. A circle k touches k_1 and the incircle of $\triangle ABC$ at points Q and P. Let R be the point of intersection of the line PQ and the bisector of $\angle BAC$ and RT be a tangent to k. Prove that $RT = \sqrt{p(p-a)}$.

2. (Russia 2000) A positive in n is called *perfect* if the sum of all its positive divisors excluding n itself, equals n. For example 6 is perfect because $6 = 1 + 2 + 3$. prove that

(a) if a perfect integer larger than 6 is divisible by 3, then it is also divisible by 9.

(b) if a perfect integer larger than 28 is divisible by 7, then it is also divisible by 49.

3. (Russia 2000) Circles ω_1 and ω_2 are internally tangent at N, with ω_1 larger than ω_2. The chords BA and BC of ω_1 are tangent to ω_2 at K and M, respectively. Let Q and P be the midpoints of the arcs AB and BC not containing the point N. Let the circumcircles of triangles BQK and BPM intersect at B and B_1. Prove that BPB_1Q is a parallelogram.

4. (Taiwan 2000) Let $A = \{1, 2, \ldots, n\}$, where n is a positive integer. A subset of A is *connected* if it is a nonempty set which consists of one element or of consecutive integers. Determine the greatest integer k for which A contains k distinct subsets A_1, A_2, ..., A_k, such that the intersection of any two distinct sets A_i and A_j is connected.

5. (Turkey 2000)

(a) Prove that for each positive integer n, the number of ordered pairs (x, y) of integers satisfying

$$x^2 - xy + y^2 = n$$

is finite and divisible by 6.

(b) Find all ordered pairs (x, y) of integers satisfying

$$x^2 - xy + y^2 = 727.$$

6. (Vietnam 2000) Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points P and Q. The common tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$ closer to P touches $\mathcal{C}_1$ at A and $\mathcal{C}_2$ at B. The tangent to $\mathcal{C}_1$ at P intersects $\mathcal{C}_2$ at E (distinct from P) and the tangent to $\mathcal{C}_2$ at P intersects $\mathcal{C}_1$ at F (distinct from P). Let H and K be two points on the rays AF and BE, respectively, such that $AH = AP$, $BK = BP$. Prove that the five points A, H, Q, K, B lie on the same circle.

7. (St. Petersburg 2000) One hundred points are chosen in the coordinate plane. Show that at most $2025 = 45^2$ rectangles with vertices among these points have sides parallel to the axes.

8. (St. Petersburg 2000) Let AA_1, BB_1, CC_1 be the altitudes of an acute triangle ABC. The points A_2 and C_2 on line A_1C_1 are such that line CC_1 bisects the segment A_2B_1 and line AA_1 bisects the segment C_2B_1. Lines A_2B_1 and AA_1 meet at K, and lines C_2B_1 and CC_1 meet at L. Prove that lines KL and AC are parallel.

9. (Korea 200) The real numbers a, b, c, x, y, z satisfy $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$. Prove that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}.$$

10. (Mongolia 2000) The bisectors of angles A, B, C of $\triangle ABC$ intersect its sides at points A_1, B_1, C_1. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then

$$\frac{BC}{AC+AB} = \frac{AC}{AB+BC} - \frac{AB}{BC+AC}.$$

SOLUTIONS

19th Iranian Mathematical Olympiad, 2001/2

1. Let p and n be natural numbers such that p is a prime and $1+np$ is a perfect square. Prove that $n+1$ is the sum of p perfect squares.

Correct solutions were received from Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Joel Tay, Ernest Chong, Charmaine Sia (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College). Their solutions are almost identical.

Suppose $1+np = k^2$ for some $k \in \mathbb{Z}$. Then $np = (k-1)(k+1)$. Since p is a prime, either $p \mid (k-1)$ or $p \mid (k+1)$. For the first case, let $sp = k-1$. Then $np = sp(sp+2)$ or

$$n+1 = s^2p + 2s + 1 = (p-1)s^2 + (s+1)^2$$

which is a sum of p squares. Similarly, for the second case, we have $tp = k+1$ and

$$n+1 = (p-1)t^2 + (t-1)^2.$$

2. Triangle ABC is acute-angled. Triangles $A'BC$, $B'AC$, $C'AB$ are constructed externally on its sides such that

$$\angle A'BC = \angle B'AC = \angle C'BA = 30°; \angle A'CB = \angle B'CA = \angle C'AB = 60°.$$

Show that if N is the midpoint of BC, then $B'N$ is perpendicular to $A'C'$.

Correct solutions were received from Joel Tay, Ernest Chong, Charmaine Sia (Raffles Junior College), Andre Kueh (Hwa Chong Junior College), Teo Weihao (National Junior College), Kenneth Tay (Anglo-Chinese Junior College) and Robert Pargeter (UK). We resent the solutions of Weihao and Kenneth.

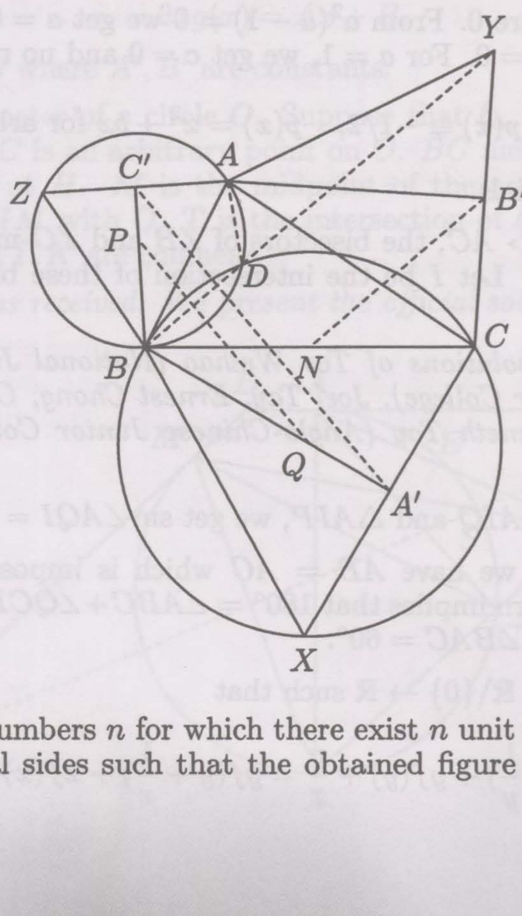
Let X, Y, Z be the reflections of C, C, A about the lines $A'B, AB', BC'$, respectively. Consider the circumcircles of $\triangle AZB$ and $\triangle CXB$ with P, Q as their respective centres. Let D be another of the intersection points. Then we have

$$\angle ADB = \angle CDB = 120° \quad \Rightarrow \quad \angle ADC = 120°.$$

Therefore $ADCY$ is cyclic since $\angle ADC + \angle AYC = 180°$. Thus

$$\angle DAC + \angle CAY + \angle DYA = \angle DYC + \angle CAY + \angle DYA = 120° = \angle ADB.$$

Hence B, D, Y are collinear. Thus $BY \perp PQ$. But $PQ \parallel A'C'$ (since $BP/BC' = BQ/BA'$) and $BY \parallel B'N$ (since N, B' are the midpoints of BC, YC, respectively). So $A'C' \perp B'N$.



3. Find all natural numbers n for which there exist n unit squares in the plane with horizontal and vertical sides such that the obtained figure has at least 3 symmetry axes.

Solved by Joel Tay, Ernest Chong (Raffles Junior College) and Andre Kueh (Hwa Chong Junior College). We present Kueh's solution.

It is not possible to have 2 parallel axes of symmetry since the number of squares is finite. Since the sides of the squares are either horizontal or vertical, the axes are either horizontal, vertical or make an angle of 45° with the horizontal. Thus two of the axes are perpendicular to each other. So $n \equiv 0, 1 \pmod 4$.

The cells of square chess boards of any dimension are examples of figures that satisfy the requirement for all such n.

4. Find all real polynomials $p(x)$ which satisfy

$$p(2p(x)) = 2p(p(x)) + 2(p(x))^2 \quad \text{for all } x \in \mathbb{R}.$$

Similar solutions by Andre Kueh (Hwa Chong Junior College), Teo Weihao (National Junior College), Joel Tay, Ernest Chong (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College).

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with $a_n \neq 0$. The terms with the highest degree in $p(2(px))$, $2p(p(x))$, $2(p(x))^2$ are, respectively, $2^n a_n^{n+1} x^{n^2}, 2a_n^{n+1}x^{n^2}, 2a_n^2 x^{2n}$. So if $n \geq 3$, we must have

$$2^n a_n^{n+1} x^{n^2} > 2a_n^{n+1} x^{n^2}$$

which is impossible. Thus $n \leq 2$. So letting $p(x) = ax^2 + bx + c$, we have

$$2a^2(a-1)x^4 + 4ab(a-1)x^3 + 2(a-1)(2ac+b^2)x^2 + 4bc(a-1)x + c(2ac - 2c - 1) \equiv 0.$$

Thus all the coefficients are 0. From $a^2(a-1) = 0$ we get $a = 0$ or $a = 1$. For $a = 0$, we get $c = -1/2, 0$ and $b = 0$. For $a = 1$, we get $c = 0$ and no restriction on b. Thus

$$p(x) \equiv 0, \quad p(x) \equiv -1/2, \quad p(x) = x^2 + bx \text{ for arbitrary } b$$

are the required polynomials.

5. In $\triangle ABC$ with $AB > AC$, the bisectors of $\angle B$ and $\angle C$ meet the opposite sides respectively at P and Q. Let I be the intersection of these bisectors. If $IP = IQ$, determine $\angle A$.

We present the similar solutions of Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Joel Tay, Ernest Chong, Charmaine Sia (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College). Also solved by Robert Pargeter (UK).

Using sine rule on $\triangle AIQ$ and $\triangle AIP$, we get $\sin\angle AQI = \sin\angle API$.

If $\angle AQI = \angle API$, we have $AB = AC$ which is impossible. Thus $\angle AQI + \angle API = 180°$. This in turn implies that $180° = \angle ABC + \angle QCB + \angle PCB + \angle ACB = 1.5(\angle ABC + \angle ACB$. So $\angle BAC = 60°$.

6. Find all functions $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ such that

$$xf(x + \frac{1}{y}) + yf(y) + \frac{y}{x} = yf(y + \frac{1}{x}) + xf(x) + \frac{x}{y}$$

for all $x, y \in \mathbb{R}\backslash\{0\}$.

We present the similar solutions of Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Joel Tay, Ernest Chong, Charmaine Sia (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College).

Define $g(x) = f(x) - x$, then after simplification we have

$$xg(x + \frac{1}{y}) + yg(y) = yg(y + \frac{1}{x}) + xg(x) \tag{1}$$

Set $y = 1$, we get

$$xg(x+1) + g(1) = g(1 + \frac{1}{x}) + xg(x) \tag{2}$$

Interchanging x with $1/x$ gives

$$\frac{1}{x}g(1 + \frac{1}{x}) + g(1) = g(x+1) + \frac{1}{x}g(\frac{1}{x}).$$

So

$$g(1 + \frac{1}{x}) = xg(x+1) + g(\frac{1}{x}) - xg(1) \tag{3}$$

Equations (2) and (3) give

$$xg(x) + g(\frac{1}{x}) = (x+1)g(1) \tag{4}$$

Setting $y = -1$, we get similarly

$$xg(x) - g(\frac{1}{x}) = -g(-1)(x-1) \tag{5}$$

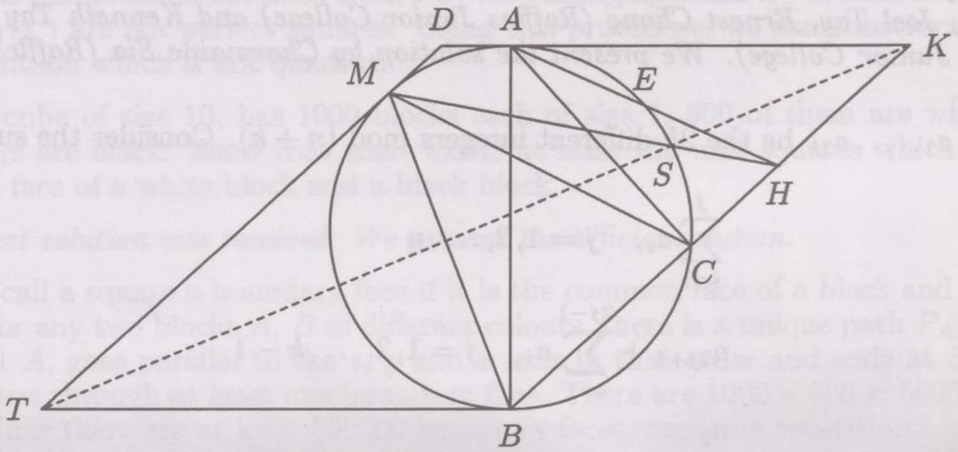
Equations (4) and (5) give

$$2xg(x) = Ax + B.$$

So $f(x) = A' + \frac{B'}{x} + x$ where A', B' are constants.

7. Let AB be a diameter of a circle O. Suppose that ℓ_a, ℓ_b are tangent lines to O at A, B, respectively. C is an arbitrary point on O. BC meets ℓ_a at K. The bisector of $\angle CAK$ meets CK at H. M is the midpoint of the arc CAB and S is another intersection point of HM with O. T is the intersection of ℓ_b and the tangent line to O at M. Show that S, T, K are collinear.

No correct solution was received. We present the official solution.



Since AB is a diameter. $\angle ACB = 90°$. Extend TM to meet ℓ_a at D. Then $\angle TMB = \angle MCB = \angle MBC$. Therefore $TD \parallel BK$. Thus $TDKB$ is a parallelogram. Let AH meet the circle at E. Then $\angle EAC = \angle EAK = \angle EBA = \angle ECA$. Therefore $AE = EC$. As $\angle ACH = 90°$, we have $AE = EC = HE$. Since $\angle AEB = 90°$, we have $AB = HB$.

Note that S, T, K are collinear iff

$$\frac{HS}{SM} = \frac{HK}{MT} \tag{1}$$

Since AH is the bisector of $\angle CAK$, we have

$$\frac{HK}{HC} = \frac{AK}{AC} = \frac{KB}{AB}, \quad \Rightarrow \quad HK = \frac{KB \cdot HC}{AB}.$$

Considering the power of H, we have

$$HS \cdot HM = HC \cdot HB, \quad \Rightarrow \quad HS = \frac{HC \cdot HB}{HM}.$$

Thus (1) is equivalent to

$$\frac{HB}{HM \cdot SM} = \frac{KB}{AB \cdot BT} \tag{2}$$

Since AB is the diameter,

$$SM = AB \sin \angle MBS = AB \sin \angle MHB.$$

The latter follows because $\angle MHB = \angle DMH = \angle MBS$. Thus (2) is equivalent to

$$AB \cdot BT = HM \cdot KB \sin MHB \quad \text{or} \quad [ABT] = [MKB]$$

where $[\cdot]$ denote the area. Since $TDKB$ is a parallelogram, we have $[MKB] = [TKB] = [ABT]$ and the proof is complete.

8. Let k be a nonnegative integer and $a_1, a_2, \ldots, a_n$ be integers such that there are at least $2k$ different integers mod $(n+k)$ among them. Prove that there is a subset of $\{a_1, a_2, \ldots, a_n\}$ whose sum of elements is divisible by $n+k$.

Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Joel Tay, Ernest Chong (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College). We present the solution by Charmaine Sia (Raffles Junior College).

Let $a_1, \ldots, a_{2k}$ be the $2k$ different integers mod $(n+k)$. Consider the sums:

$$\sum_{i=1}^{j} a_i, \quad j = 1, 2, \ldots, n$$

$$a_{2j+1} + \sum_{i=1}^{2j-1} a_i, \quad j = 1, 2, \ldots, k-1$$

$$a_2$$

There are altogether $n+k$ sums. If all the sums are distinct mod $(n+k)$, then one of them is 0 mod $(n+k)$ and we are done. Now suppose that there are two sums which are congruent mod $(n+k)$. Note that if two sums have the same number of terms, they cannot be congruent mod $(n+k)$. The two sums must have different number of terms. In this case, their difference is the sum of a subset and is 0 mod $(n+k)$.

9. Consider a permutation $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$. We call this permutation quadratic if there exists at least one perfect square among the numbers $a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n$. Find all natural numbers n such that every permutation of $\{1, 2, \ldots, n\}$ is quadratic.

We present solution of Andre Kueh (Hwa Chong Junior College) and Ernest Chong (Raffles Junior College).

If $n(n+1)/2$ is a square, then clearly every permutation of $\{1, 2, \ldots, n\}$ is quadratic. We shall first characterize such n. Let $n(n+1)/2 = m^2$. So

$$(2n+1)^2 - 2(2m)^2 = 1.$$

Set $x = 2n+1$, $y = 2m$, we get $x^2 - 2y^2 = 1$, which is a Pell's equation with primitive roots $x_0 = 3$, $y_0 = 2$. So if (x, y) is a solution then

$$x + \sqrt{2}y = (3 + 2\sqrt{2})^m, \quad m = 1, 2, \ldots$$

Thus

$$n = \frac{(3+2\sqrt{2})^m + (3-2\sqrt{2})^m - 2}{4}, \quad m = 1, 2, \ldots.$$

We now show that no other natural numbers have this property.

Consider a permutation $A = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ where $n(n+1)/2$ is not a square,. Let $s_k(A) = \sum_{i=1}^k a_i$. Let I be the identity permutation, i.e., $a_i = i$ for all i. Then $s_k(I) = k(k+1)/2$. Suppose $k(k+1) = m^2$. Then $k < 2m$. So

$$\frac{(k+1)(k+2)}{2} = m^2 + k + 1 < m^2 + 2m + 1 = (m+1)^2.$$

If I' is the permutation obtained from I by interchanging k and $k+1$, then both $s_k(I')$ and $s_{k+1}(I')$ are not perfect squares. Using this procedure, we can construct from I a permutation which is not quadratic.

10. A cube of size 10, has 1000 blocks each of size 1, 500 of them are white and the others are black. Show that there exists at least 100 unit squares which are the common face of a white block and a black block.

No correct solution was received. We present the official solution.

We call a square a boundary face if it is the common face of a black and a white block. For any two blocks A, B of different colours, there is a unique path P_{AB} which begins at A, goes parallel to the z, y and x axes in that order and ends at B. Each P_{AB} passes through at least one boundary face. There are $1000 \times 500 = 500000$ such paths. Thus there are at least 500000 boundary faces, counting repetition.

Consider an arbitrary square S parallel to the xy plane. Assume that there are k blocks below it. For a path P_{AB} to pass through S, A must be directly below S and B any block above the plane containing S or A is directly above S and B any block below the plane containing S. In the former, there are $k \times (100 \times (10 - k))$ paths. In the latter, there are $(10 - k) \times (100 \times k)$ paths. Thus each boundary face is counted $200k(10 - k) \leq 5000$ times. Thus the number of boundary faces is $\geq 500000/5000 = 100$.

Slovenia Mathematical Olympiad, 2002

Selected problems from the final round.

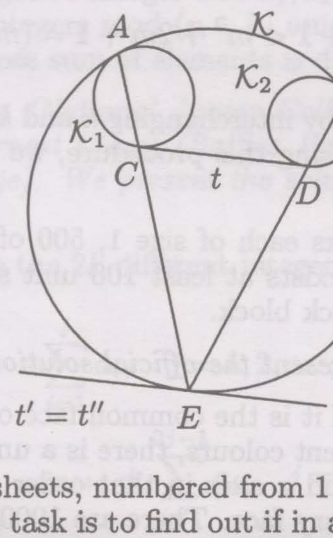
1. Let $\mathcal{K}$ be a circle in the plane, and $\mathcal{K}_1$, $\mathcal{K}_2$ be two disjoint circles inside $\mathcal{K}$ and touching $\mathcal{K}$ at A, B, respectively. Let t be the common tangent line of $\mathcal{K}_1$ and $\mathcal{K}_2$ at C and D, respectively such that $\mathcal{K}_1$, $\mathcal{K}_2$ are on the same side of t while the centre of $\mathcal{K}$ is on the opposite side. Denote by E the intersection of lines AC and BD. Prove that E lies on $\mathcal{K}$.

Solved by Robert Pargeter (UK), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Andre Kueh (Hwa Chong Junior College), Teo Weihao (National Junior College). We present the similar solutions of Joel, Charmaine, Andre and Weihao.

Let h be the homothety with centre A that maps $\mathcal{K}_1$ to $\mathcal{K}$. Extend AC to meet $\mathcal{K}$ at E'. Then h maps t to t' which is the tangent of K at E'. Note that $t \parallel t'$.

Extend BD to meet $\mathcal{K}$ at E''. Similarly, the homothety h' with centre B that takes $\mathcal{K}_2$ to K also take t to t'', the tangent of $\mathcal{K}$ at E''. Since $t \parallel t''$, we have $t' = t''$. Thus $E' = E'' = E$.

(Note: Pargeter pointed that the location of O with respect to t is not important. The result remains true if all the three centres are on the same side of t. This is also evident from the solution given.)



2. There are $n \geq 3$ sheets, numbered from 1 through n. The sheets are then divided into two piles and the task is to find out if in at least one pile two sheets can be found such that the sum of the corresponding numbers is a perfect square. Prove that

(a) if $n \geq 15$, then two such sheets can always be found;

(b) if $n \leq 14$, then there is a division in which two such sheets cannot be found.

Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College). We present their similar solutions.

(a) Assume, on the contrary, that the sum of any pair of numbers in any of the piles is not a perfect square. Let the piles be A, B. Without loss of generality, assume that $1 \in A$. Then

$$1 \in A \Rightarrow 3 \in B \Rightarrow 6 \in A, \Rightarrow 10 \in B \Rightarrow 15 \in A.$$

This leads to a contradiction.

(b) The following is an example.

$$A = \{1, 2, 4, 6, 9, 11, 13\}, B = \{3, 5, 7, 8, 10, 12, 14\}.$$

3. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(2002)) = 17, \quad f(mn) = f(m)f(n), \quad \text{and} \quad f(n) \leq n$$

for every $m, n \in \mathbb{N}$?

Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College). We present their similar solutions.

Since $2002 = 2 \times 7 \times 11 \times 13$, we have

$$f(f(2002)) = f(f(2)f(7)f(11)f(13)) = f(f(2))f(f(7))f(f(11))f(f(13)) = 17.$$

Thus $f(f(2)), f(f(7)), f(f(11)), f(f(13))$ is a permutation of 1,1,1,17. But $f(f(x)) \leq f(x) \leq x$. So the above cannot happen. Thus the function cannot exists.

4. Let $S = \{a_1, \ldots, a_n\}$ where a_i are different positive integers. The sum of all numbers of any proper subset of the set S is not divisible by n. Prove that the sum of all numbers of the set S is divisible by n.

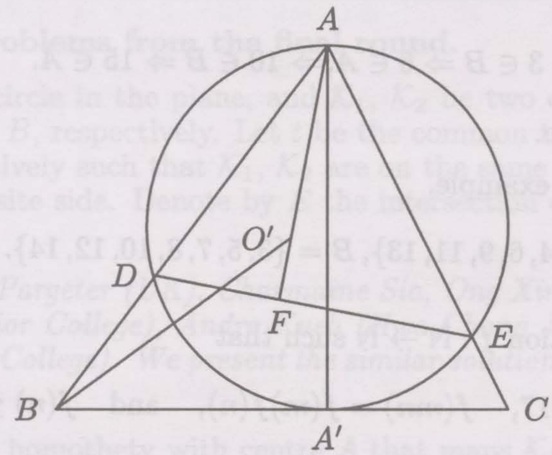
Solved by Andre Kueh (Hwa Chong Junior College), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College). We present their similar solutions.

Let $S_i = a_1 + \cdots + a_i$, $i = 1, \ldots, n$. For $1 \leq i < j \leq n$, $S_j - S_i$ is the sum of a proper subset of S. Thus $S_j - S_i \not\equiv 0 \pmod{n}$. Hence $S_i, \ldots, S_n$ are distinct (mod n) and so one of them is divisible by n. Since $S_i \not\equiv 0 \pmod{n}$ for $1 \leq i \leq n-1$, we have $n \mid S_n$.

5. Let A' be the foot of the altitude on the side BC of $\triangle ABC$. The circle with diameter AA' intersects the side AB at the points A and D, and the side AC at the points A and E. Prove that the circumcentre of $\triangle ABC$ lies on the line determined by the altitude on the side DE of $\triangle ADE$.

Solved by Charmaine Sia (Raffles Junior College), Teo Weihao (National Junior College) and Robert Pargeter (UK). We present the similar solutions of Weihao and Charmaine.

Let F be the foot of the altitude from A onto DE, O be the midpoint of AA' and O' be the circumcentre of $\triangle ABC$. To prove that O' lies on AF, we merely need to show that $\angle BAF = 90° - \angle ACB$. This follows readily from $\angle ADE = \angle AA'E = \angle ACB$.



6. In the cellar of a castle 7 dwarfs protect their treasure. The treasure is behind 10 doors and every door has 3 locks. All locks are different from each other. Every dwarf has the keys for some locks. Any four dwarfs together have keys for all the locks. Prove that there exist three dwarfs who together have the keys for all the locks.

Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College) who gave similar solutions.

Consider each lock in turn. If there are less than 4 dwarfs with keys to the lock, then there exist 4 dwarfs who do not have the key to the lock which gives a contradiction. Thus there are at least 4 dwarfs with the key to the lock. Hence there is at most one choice of 3 dwarfs with no key to the lock. But there are $\binom{7}{3} = 35$ choices of 3 dwarfs but only 30 locks so there is at least one choice of 3 dwarfs who have keys to all the locks.